Testing structural hypotheses in a multivariate cointegration analysis of the PPP and the UIP for UK*

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The paper develops some new tests for structural hypotheses in the framework of a multivariate error correction model with Gaussian errors. The tests are constructed by an analysis of the likelihood function and motivated by an empirical investigation of the PPP relation and the UIP relation for the United Kingdom.

Three types of tests are discussed. First we consider the same linear restrictions on all cointegration relations, then we consider the hypothesis that certain relations are assumed to be cointegrating, and finally we formulate a general hypothesis that contains the previous ones. This hypothesis can be expressed by the condition that some of the cointegrating relations are subject to given linear restrictions, while others are unconstrained.

1. Introduction

The methodological purpose of this paper is to develop some tests for structural hypotheses on the cointegrating relations in a multivariate error correction model. We consider a VAR model in levels, under the assumption of cointegration. This model is used to describe the statistical variation of the data without imposing a priori restrictions implied by the economically interesting relations that motivated the empirical analysis. Instead the parametric formulation allows us to formulate a set of structural economic hypotheses as statistical hypotheses concerning the cointegrating relations. This again allows for a likelihood analysis if we assume Gaussian errors, and the tests proposed are likelihood ratio tests. We consider three types of hypotheses. First the hypothesis of linear restrictions on the cointegration

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relations. This type of hypothesis was discussed in Johansen (1988) and Johansen and Juselius (1990). Next we consider the hypothesis that some cointegration relations are known, while others are unrestricted. Finally we formulate the general hypothesis that some cointegrating relations are restricted by linear restrictions while the others are unrestricted.

The empirical purpose of the paper is to investigate international transmission effects between countries through determination of exchange rate, interest rates, and prices, as assumed by the purchasing power parity and the uncovered interest rate parity relation. This has been the focus of interest in a vast number of empirically oriented papers; see, for instance, Adler and Lehmann (1983), Baillie and Selove (1987), Corbæs and Ouliaris (1988), Edison and Klovland (1987), Hakkio (1984), and Schotman (1989). Generally the empirical evidence of these fundamental relations has been weak [see, e.g., Frenkel (1981) and Dornbusch (1989)]. Here we will try to suggest possible reasons why so many studies have failed in this respect. This will point to the importance of considering the interaction between exchange rates, interest rates, and prices in the goods and the asset markets in a simultaneous model as well as the importance of distinguishing between short-run and long-run effects.

We will analyze time series data from the UK economy using an econometric modelling approach which differs from the standard ones in two important ways. Firstly, we will analyze the data in a full system of equations, thus allowing for possible interactions in the determination of prices, interest rates, and exchange rates. This would eliminate the single-equation bias likely to have affected many of the previous studies. Secondly, we will adopt a model specification that explicitly allows for different short-run and long-run dynamics using recent results on nonstationary time series. The distinction between short-run and long-run effects is crucial in this empirical problem, since two different types of markets are involved. In the goods market arbitrage is costly, whereas it is much less so in the asset market. Consequently one can assume that exchange rates are affected by short-run fluctuations arising from highly volatile asset markets and by long-run effects from interrelated goods markets.

The econometric analysis is based on the multivariate cointegration model which is well designed for this type of empirical work by the explicit classification into nonstationary and stationary components providing an interpretation in terms of the dynamics of long-run and short-run effects.

The paper is organized as follows: In section 2 we discuss briefly the economic background and imbed the economic problem in the vector autoregressive model with Gaussian errors. In section 3 the multivariate cointegration model is discussed in the light of the empirical problem. In section 4 the empirical analysis of the basic unrestricted VAR model is reported and the adequacy of the Gaussian assumption is tested. Tests for weak exogeneity
w.r.t. the long-run parameters are reported. In section 5 we present three types of hypotheses on the cointegrating relations corresponding to various interesting economic hypotheses. For each hypothesis we describe the calculation of the test statistic and the degrees of freedom. For the asymptotic theory we refer to previously published work. Section 6 contains some concluding remarks.

2. The economic framework

The most popular models that have been applied for exchange rate determination include the flexible price monetary model of Frenkel (1976) and the overshooting monetary model of Dornbusch (1976). However, the empirical success of these models has not been too convincing. For the interest rate determination various versions of the uncovered real or nominal interest rate parity have been the standard reference. Prices in open economies are usually assumed to be determined by various versions of the purchasing power parity relation. The link between prices, interest rates, and exchange rates can be found in the real interest rate differential model of Frankel (1979) relating an interest rate differential between two countries to (i) a possible covered interest rate differential, (ii) an expected change in exchange rates, and (iii) an expected change in purchasing power parity between the countries.

Here we will focus on the absolute version of the purchasing power parity (PPP) relation and the uncovered interest rate parity (UIP) relation. PPP relates nominal exchange rates to the price levels in two countries, $p_1 - p_2 = e_{12}$, where $p_1$ and $p_2$ indicate the price level in country 1 and 2, respectively, and $e_{12}$ is the exchange rate denominated in the currency of country 1 and (UIP) relates interest rates of two countries to expected changes in exchange rates, $i_1 - i_2 = e_{12}^e - e_{12}$. A superscript $e$ indicates expectation and all variables are assumed to be in logarithms. The absolute version of the PPP hypothesis as a description of the interdependence of markets and prices is usually assumed to be valid only as a long-run relation, if valid at all. The popular competing hypothesis that real exchange rates follow a martingale process has found some empirical support [see, e.g., Adler and Lehmann (1983) and Hakkio (1984)]. Whatever the true case, there can hardly be any doubt that if the PPP holds as a long-run relation, the speed of adjustment has to be very slow due to costly information gathering, product heterogeneity, government-imposed barriers to trade, etc.¹

¹There are however other explanations why the PPP often has proven an empirical failure related to the measurement of directly comparable aggregate price indices, as for instance differences between countries in productivity growth and in the proportion of tradeables to nontradeables.
The uncovered interest rate parity describes a forward-looking market clearing mechanism which, if the widely used assumption of efficient asset markets is correct, would be relatively fast. However, most empirical works on UIP have been disappointing by not being able to verify this relation as an immediate market clearing mechanism. Instead empirical evidence seems to suggest that UIP holds as a long-run relation, probably due to the existence of a stationary time-dependent risk premium. Therefore we will here assume the less restrictive hypothesis that the UIP is a stationary process and that both prices and interest rates adjust to realized deviations from this parity. Based on the above arguments we consider a model formulation that does not directly impose the UIP and the PPP relation but, more indirectly, assumes a tendency in the market to react to deviations from these relations. These considerations motivate a model specification of the error-correction type:

**Prices**
\[
\Delta p_{1t} = f_{p1}(\Delta X) + \gamma_1(\text{ecm}_{\text{ppp}})_{t-1} + \gamma_2(\text{ecm}_{\text{uip}})_{t-1} + \varepsilon_{p1t},
\]
\[
\Delta p_{2t} = f_{p2}(\Delta X) + \gamma_3(\text{ecm}_{\text{ppp}})_{t-1} + \gamma_4(\text{ecm}_{\text{uip}})_{t-1} + \varepsilon_{p2t},
\]

**Exchange rates**
\[
\Delta e_{12t} = f_e(\Delta X) + \gamma_5(\text{ecm}_{\text{ppp}})_{t-1} + \gamma_6(i_1 - i_2)_{t-1} + \varepsilon_{et},
\]

**Interest rates**
\[
\Delta i_{1t} = f_{i1}(\Delta X) + \gamma_7(\text{ecm}_{\text{ppp}})_{t-1} + \gamma_8(\text{ecm}_{\text{uip}})_{t-1} + \varepsilon_{it},
\]
\[
\Delta i_{2t} = f_{i2}(\Delta X) + \gamma_9(\text{ecm}_{\text{ppp}})_{t-1} + \gamma_{10}(\text{ecm}_{\text{uip}})_{t-1} + \varepsilon_{i2t}.
\]

Here \( X = [x_1, \ldots, x_p] \) is a vector of explanatory variables and \( f_m(\Delta X) \) indicates a linear function of \( \Delta X_t, \Delta X_{t-1}, \ldots \), in which \( \Delta x_{mt} \) is excluded and \( m = p_1, p_2, e_{12}, i_1, i_2 \) and where \( \text{ecm}_{\text{ppp},t-1} = (p_1 - p_2 - e_{12})_{t-1} \) and \( \text{ecm}_{\text{uip},t-1} = (\Delta e_{12})_t - (i_1 - i_2)_{t-1} \). No assumption is made at this stage on the specific form of the short-run dynamics. In particular note that even if \( \Delta e_{12t} \) is restricted by the coefficient in the UIP relation, the inclusion of \( \Delta e_{12t} \) unrestrictedly in \( f_m(\Delta X) \) allows exchange rate effects through other channels than the UIP. Thus in the above form the model allows for other explanations in the short run, but presumes PPP and UIP to determine the long-run steady-state solution, which could be derived by assuming constant nominal growth of prices and consequently constant exchange rates and interest rates. It is then obvious that the interesting long-run relations are the PPP and the interest rate differential (since in steady-state \( \Delta e_{12} = 0 \)).
Since time series data on prices, exchange rates, and interest rates in levels must be considered nonstationary, we will use recent advances in the analysis of nonstationary time series [Johansen (1988), Johansen and Juselius (1990)] to analyze whether there exist stationary linear relations between the levels of the variables, and if this is the case, whether the unrestricted result is consistent with the hypothetical long-run relations.

The purpose of specifying the error-correction system of equations above is also to illustrate that if PPP is a valid description of international price dependence, information on this relation can be found in all equations and traditional single-equation analysis would be inefficient. In that case standard inference is no longer valid due to the nonstationarity of prices [see, e.g., Johansen (1991a)]. The same argument is clearly true for the interest rate differential as well. Note also that by allowing for short-run effects arising from changes in prices, interest rates, and exchange rates in all equations of the system, it is likely that the variance of the residuals is considerably reduced compared to an analysis of only prices and exchange rates. This should increase the efficiency of inference, possibly decisively.

In the empirical analysis described in the next section, the reduced form of the error-correction system will be investigated with the lag length long enough for the residuals to be uncorrelated. The hypothetical parameter restrictions implied by the long-run PPP and the interest rate differential will not be imposed but instead tested for data admissibility with the unrestricted cointegration space based on the multivariate cointegration model [Johansen and Juselius (1990)]. First we ask whether the cointegration space contains the purchasing power parity restriction for all cointegration vectors. Then we ask whether the PPP relation is stationary by itself without involving the other variables of the system. We will also consider the hypothesis whether some linear combination of \( p_1, p_2, \) and \( e_{12} \) is stationary. This could be of interest if the previous hypotheses were rejected.

### 3. The statistical model

We will now turn to the basic model, the five-dimensional vector autoregressive model with Gaussian errors,

\[
X_t = A_1 X_{t-1} + \cdots + A_k X_{t-k} + \mu + \Psi D_t + \varepsilon_t, \quad t = 1, \ldots, T, \tag{1}
\]

where \( X_t = [p_1, p_2, e_{12}, i_1, i_2] \), as defined above, \( X_{t-k+1}, \ldots, X_0 \) are fixed, \( \varepsilon_1, \ldots, \varepsilon_T \) are i.i.d. \( N(0, \Sigma) \), and \( D_t \) are centered seasonal dummies. Since we want to distinguish between stationarity by linear combinations and by differencing we write the model in error correction form:

\[
\Delta X_t - \Gamma_1 \Delta X_{t-1} - \cdots - \Gamma_{k-1} \Delta X_{t-k+1} + \Pi X_{t-k} + \mu + \Psi D_t + \varepsilon_t, \quad t = 1, \ldots, T, \tag{2}
\]
and assume in the following hypothesis:

$$H_1(r): \Pi = \alpha \beta',$$

(3)

where $\alpha$ and $\beta$ are $p \times r$ matrices. The hypothesis $H_1(r)$ is the hypothesis of reduced rank of $\Pi$ implying that under certain conditions [see Johansen (1989)] the process $\Delta X_t$ is stationary, $X_t$ is nonstationary, but also that $\beta'X_t$ is stationary. Thus we can interpret the relations $\beta'X_t$ as the stationary relations among nonstationary variables, i.e., as cointegrating relations.

However, the real importance of model formulation (2) with the hypothesis of cointegration (3) is that it allows the precise formulation of a number of interesting economic hypotheses in such a way that they can be tested. This will be demonstrated in section 5, where we consider the testing of three different types of structural hypotheses.

It was shown in Johansen (1988) and Johansen and Juselius (1990) how one calculates the maximum likelihood estimator in the multivariate cointegration model. In section 5 it will be shown that simple modifications of this procedure yield the estimates and test statistics under the structural hypotheses discussed above. Below we give a brief description of the estimation procedure to introduce the necessary notation and the most important concepts.

The likelihood function is first concentrated with respect to the parameters $\Gamma_1, \ldots, \Gamma_{k-1}, \mu$, and $\nu$ by regressing $\Delta X_t$ and $X_{t-k}$ on $\Delta X_{t-1}, \ldots, \Delta X_{t-k+1}, 1$, and $D_t$. This defines the residuals $R_{0t}$ and $R_{kt}$ and the residual product moment matrices

$$S_{ij} = T^{-1} \sum_{t=1}^{T} R_{it} R'_{jt}, \quad i, j = 0, k.$$

(4)

The concentrated likelihood function has the form of a reduced rank regression,

$$R_{0t} = \alpha \beta' R_{kt} + \text{error}. $$

(5)

For fixed $\beta$ (5) can be solved for $\alpha$ by regression

$$\hat{\alpha} (\beta) = S_{0k} \beta (\beta' S_{kk} \beta)^{-1}, $$

(6)

and $\beta$ is determined by solving the eigenvalue problem:

$$|\lambda S_{kk} - S_{k0} S_{00}^{-1} S_{0k}| = 0. $$

(7)

This has the solutions $\hat{\lambda}_1 > \cdots > \hat{\lambda}_p > 0$ with the corresponding eigenvectors
\[ \hat{\beta} = (\hat{\epsilon}_1, \ldots, \hat{\epsilon}_r), \] (8)

which with the above normalization gives

\[ \hat{\alpha} = S_{0k} \hat{\beta} = (\hat{\omega}_1, \ldots, \hat{\omega}_r). \] (9)

From the property \( \hat{\alpha}^T S_{kk} \hat{\alpha} = I \) it follows that the unrestricted estimate \( \hat{\alpha}' S_{kk} \hat{\alpha} = I \) can be written \( S_{0k} \hat{\alpha}' \hat{\alpha} = \sum_{i=1}^r \hat{\epsilon}_i \hat{\beta}_i + \sum_{i=r+1}^p \hat{\omega}_i \hat{\beta}_i \). It follows from the definition of \( \hat{\epsilon}_i \) as eigenvectors that

\[
\left( \begin{array}{c}
\hat{\omega}_{r+1}, \ldots, \hat{\omega}_p
\end{array} \right) S_{00}^{-1} \left( \begin{array}{c}
\hat{\omega}_{r+1}, \ldots, \hat{\omega}_p
\end{array} \right) \\
= \left( \begin{array}{c}
\hat{\epsilon}_{r+1}, \ldots, \hat{\epsilon}_p
\end{array} \right) S_{k0} S_{00}^{-1} S_{0k} \left( \begin{array}{c}
\hat{\epsilon}_{r+1}, \ldots, \hat{\epsilon}_p
\end{array} \right) \\
= \text{diag} \left( \lambda_{r+1}, \ldots, \lambda_p \right). \] (10)

The maximized likelihood function is found to be

\[ L_{\text{max}}^{-2/T} = |S_{00}| \prod_{i=1}^r (1 - \hat{\lambda}_i), \]

and the likelihood ratio test for the hypothesis \( \mathcal{H}_1(r) \) in the full VAR model (2), \( \mathcal{H}_0 \), is given by

\[ -2 \ln Q(\mathcal{H}_1(r)|\mathcal{H}_0) = -T \sum_{i=r+1}^p \ln (1 - \hat{\lambda}_i), \] (11)

which is called the trace statistics. An alternative test statistic called the \( \lambda_{\text{max}} \) statistic is based on the comparison of \( \mathcal{H}_{r-1}(r) \) against \( \mathcal{H}_1(r) \):

\[ -2 \ln Q(\mathcal{H}_{r-1}(r)|\mathcal{H}_1(r)) = -T \ln (1 - \hat{\lambda}_r). \] (12)

Thus the test statistic (11) which, using (10), is approximately \( T \sum_{i=r+1}^p \hat{\lambda}_i = T \sum_{i=r+1}^p \hat{\epsilon}_i' S_{kk}^{-1} \hat{\omega}_i \) and is really measuring the size of the coefficients of the supposedly nonstationary components \( \hat{\epsilon}_i' X_{t-k} \) in the full regression model. A full discussion of the test statistics is given in Johansen and Juselius (1990) together with the necessary tables for making inference. The asymptotic distributions are discussed in Johansen (1991b).
4. The empirical results of the cointegration analysis

4.1. The data

Before defining the empirical model, the data given as quarterly observations will be briefly described. The basic variables of interest are:

\[ p_1 = \text{UK wholesale price index}, \]
\[ p_2 = \text{trade weighted foreign wholesale price index}, \]
\[ e_e = \text{UK effective exchange rate}, \]
\[ i_1 = \text{three-month treasury bill rate in the UK}, \]
\[ i_2 = \text{three month Eurodollar interest rate}. \]

All variables are in logarithms and the sample period is 1972.1 to 1987.2, thus covering the post-Bretton Woods floating exchange rate system. The graphs of the differenced data are given in appendix 1 and illustrate the large fluctuations in the data. These are partly the result of the two oil crises and the great turbulence in the exchange market during this period. We indicate below some major events of importance:

1973.3–4: first oil crisis,
1979: tight monetary policy introduced by Mrs. Thatcher,
1979: abandonment of the exchange control in the UK,
1979: second oil crisis and UK becoming an oil producer,
1980: depository institutions deregulation and monetary control act in the USA,

The last two events have strongly influenced interest rate determination outside the borders of USA. Altogether the above events are likely to have changed some parameters of the model, motivating some care when interpreting the empirical results. However, we believe that the short-run parameters are more likely to change as a result of interventions in the economy, rather than the more fundamental long-run parameters. In this application we will explicitly account for the effect of the changes in the world oil price, but leave the effects of other interventions to be accounted for by a general specification of the short-run dynamics.

4.2. The basic empirical model

Model (2) together with the reduced rank hypothesis (3) is the starting point of the empirical analysis. However, if there are linear trends in the data, both the estimation procedure and the rank inference will differ compared to the case with no linear trends. In the former case the constant term in (2) can be partitioned into \( \mu = \alpha \beta_0 + \alpha \gamma, \) where \( \beta_0 \) is a \( r \times 1 \) vector
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of intercepts in the cointegrating relations, \( \alpha_\perp \) is a \( p \times (p - r) \) matrix of full rank orthogonal to the columns of \( \alpha \), and \( \gamma \) is a \( (p - r) \times 1 \) vector of linear trend slopes. Inspection of the graphs of the differenced data in appendix 1 clearly indicated that the price series have a linear trend consistent with the steady-state assumption of constant nominal price growth discussed above. Therefore model (2) will be estimated without imposing linear restrictions on \( \mu \). Since the empirical evidence of linear trends is quite strong, the alternative hypothesis \( \alpha_\perp \gamma = 0 \) has not been tested. This test has been described in Johansen and Juselius (1990) as well as the estimation procedure under the restriction \( \alpha_\perp \gamma = 0 \). Note that under the assumption of common linear trends these will only appear in the constant term of the model, since they are supposed to cancel in the cointegration relations. Explicit inclusion of a linear time trend in the cointegration relations would therefore imply that there is some linear growth which our model cannot account for given the chosen data set.

Another question of interest is whether prices are second-order nonstationary and therefore should be differenced twice. The graphs of the differenced price series indicate nonstationarity in the variance rather than in the mean, meaning that differencing is not the appropriate way of removing it. Actually model (2) was first estimated for \( k = 2 \), but the residuals did not pass the normality test due to excess kurtosis. Large residuals were found to coincide with large changes in the oil price. This problem was tackled by conditioning on the changes in the oil price, the movements of which were considered a hypothetical source to the nonstationarity in variance. Since the oil price can be assumed exogenously determined in this system we have reformulated (2) in the following way:

\[
\Delta X_t = \Gamma_1 \Delta X_{t-1} + C_0 \Delta x_{6t} + C_1 \Delta x_{6t-1} + \Pi X_{t-2} + \mu + \Psi D_t + \epsilon_t, \quad t = 1, \ldots, T, \tag{13}
\]

where \( x_{6t} \) is the logarithm of the world oil price at time \( t \) and \( X_t \) is defined as before.

The misspecification tests for this model are reported in table 1 below. The results are now more satisfactory, although there is still indication of excess kurtosis in eq. 2 and eq. 5, causing the Jarque–Bera test statistic for normality to become significant. This is not surprising since the combined price index, \( x_2 \), and the Eurodollar rate, \( x_5 \), were chosen to explain the variation in UK price, exchange rate, and interest rates but not vice versa, meaning that the selected variable set is probably not sufficient to account for the variation in \( x_2 \) and \( x_5 \). The variable \( x_2 \) and \( x_5 \) might well be weakly exogenous for the long-run parameters of interest, which would make the deviation from normality less important. The weak exogeneity is a testable
Table 1
Residual misspecification tests in model (13).

<table>
<thead>
<tr>
<th>Eq.</th>
<th>Standard deviation</th>
<th>Skewness</th>
<th>Excess kurtosis</th>
<th>Normality test $\chi^2(2)$</th>
<th>Autocorr. test $\chi^2(20)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.007</td>
<td>0.29</td>
<td>1.27</td>
<td>4.84</td>
<td>6.09</td>
</tr>
<tr>
<td>2</td>
<td>0.007</td>
<td>0.28</td>
<td>2.16</td>
<td>12.44</td>
<td>9.59</td>
</tr>
<tr>
<td>3</td>
<td>0.030</td>
<td>0.30</td>
<td>0.17</td>
<td>0.95</td>
<td>13.54</td>
</tr>
<tr>
<td>4</td>
<td>0.011</td>
<td>0.58</td>
<td>0.25</td>
<td>3.55</td>
<td>9.11</td>
</tr>
<tr>
<td>5</td>
<td>0.013</td>
<td>-0.51</td>
<td>3.76</td>
<td>37.95</td>
<td>16.41</td>
</tr>
</tbody>
</table>

hypothesis as will be shown below. Here we conclude that the residuals from UK prices, exchange rates, and interest rates can be assumed to follow a Gaussian process and the residuals from the combined prices and the Eurodollar interest rates follow an innovation process.

The graphs of the residuals $\hat{e}_{it}$ and the first differences $\Delta x_{it}$ ($i = 1, \ldots, 5$) are presented in appendix 1. Note how well the large fluctuations in the original data have been accounted for by the chosen information set.

The sensitivity of the analysis to the choice of lag length and error distribution can be investigated by Monte Carlo experiments; see Gonzalo (1989). It follows from the asymptotic analysis [see Johansen (1988, 1991b)] that the limit results hold as long as the errors admit a central limit theorem, in the sense that the cumulative sums converge to Brownian motions. Thus the assumption of Gaussian errors is only applied in order to derive the test statistics and estimators.

4.3. Testing for reduced rank

The test of the rank of $\Pi$ is performed using the two likelihood ratio tests (11) and (12), the results of which are presented in table 2 below.

Based on the $\lambda_{\text{max}}$ test statistic the hypothesis of no cointegration cannot be rejected at the standard 5% level, whereas the trace statistic would lead us

Table 2
Tests of the cointegration rank.

<table>
<thead>
<tr>
<th>i</th>
<th>$\hat{\lambda}_i$</th>
<th>$-T \ln(1 - \hat{\lambda}_i)$</th>
<th>$\hat{\lambda}_{\text{max}}(0.95)$</th>
<th>$-T \sum \ln(1 - \hat{\lambda}_i)$</th>
<th>$\hat{\lambda}_{\text{trace}}(0.95)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.407</td>
<td>31.33</td>
<td>33.18</td>
<td>80.75</td>
<td>68.91</td>
</tr>
<tr>
<td>2</td>
<td>0.285</td>
<td>20.16</td>
<td>27.17</td>
<td>49.42</td>
<td>47.18</td>
</tr>
<tr>
<td>3</td>
<td>0.254</td>
<td>17.59</td>
<td>20.78</td>
<td>29.26</td>
<td>29.51</td>
</tr>
<tr>
<td>4</td>
<td>0.102</td>
<td>6.48</td>
<td>14.03</td>
<td>11.67</td>
<td>15.20</td>
</tr>
<tr>
<td>5</td>
<td>0.083</td>
<td>5.19</td>
<td>3.96</td>
<td>5.19</td>
<td>3.96</td>
</tr>
</tbody>
</table>
to accept two cointegration vectors. To make the decision still more difficult, note that \( \hat{\lambda}_3 \approx \hat{\lambda}_2 \) suggesting that possibly the third eigenvector should be considered among the stationary vectors. The interpretation of the eigenvalues as given by (10) is also useful for the understanding of the behavior of the test statistics. Consider the second cointegration vector \( \beta_{0.2} \) and the corresponding values of \( \alpha_{i2} \) in table 3 below which, for \( i = 1, 2, 3 \), are approximately zero. The value of \( \hat{\lambda}_2 \) is thus affected by the number of zero and nonzero coefficients in the second column, indicating that a low \( \hat{\lambda}_2 \) value might be the result of many \( \alpha_{i2} = 0 \) for that particular \( \beta_{0.2} \).

The test results thus illustrate the fact that the two test procedures do not necessarily give the same result and consequently the ambiguity when choosing the number of cointegrating vectors. Basically this ambiguity is due to the low power in cases when the cointegration relation is quite close to the nonstationary boundary [Johansen (1991c)]. When considering that the null hypothesis of a unit root is not always reasonable from an economic point of view this can be a serious problem which has recently been discussed by a.o. Schotman and van Dijk (1989). This problem is often present in empirical work when the speed of adjustment to the hypothetical equilibrium state is very slow for instance due to regulations, high adjustment costs, and other short-run effects which tend to keep the process off the equilibrium path. In some applications economic theory might be helpful by suggesting the number of common trends driving the system and consequently the rank \( r \); see, for instance, Baillie and Bollerslev (1989) for an application to seven daily exchange rates.

In the present application the final determination of the number of cointegration vectors is based both on the result of formal testing and the interpretability of the obtained coefficients as well as the graphs. In appendix 2 the graphs of \( \ddot{v}_tX_t \) and \( \ddot{v}_tR_{kt} \) are given. The estimated values \( \ddot{v}_t \) are given in table 3 below and \( R_{kt} \) is defined in section 3. If \( r = 2 \), we would expect the first two processes to look stationary, albeit not like white noise processes. The ordering of the relations based on the ordering of \( \hat{\lambda}_i \) means that the first relation, \( \beta_1X_t \), is most correlated with the stationary part of the process \( \Delta X_t \) when corrected for the lagged values of the differences, and the second is the next most correlated, etc. The graphs of the cointegrating relations corrected for the short-run dynamics, \( \beta_1R_{ki} \), look much more satisfactory in this respect compared to the graphs of \( \beta_1X_t \). This gives an illustration of an important property of this model, namely its ability to describe an inherent tendency to move towards the equilibrium states, without necessarily ever reaching it because of frequent and often large shocks pushing it away from the equilibrium path. In that sense the graphs \( \beta_1X_t \) describe the actual deviation from the equilibrium path as a function of short-run effects, whereas \( \beta_1R_{ki} \) describes the adjustment path corrected for the short-run dynamics of the model. If the short-run dynamics can be expected to be
The estimated eigenvectors of (7) partitioned into the stationary components (β_{0,1}, β_{0,2}) and their weights (α_{0,1}, α_{0,2}) together with the remaining eigenvectors (β_3, β_4, β_5) and the corresponding weights (ω_3, ω_4, ω_5) defined as \( \hat{w}_i = S_{0i}v_i \); see (9).

<table>
<thead>
<tr>
<th>Eigenvectors</th>
<th>( \hat{\beta}_{0,1} )</th>
<th>( \hat{\beta}_{0,2} )</th>
<th>( \hat{\beta}_3 )</th>
<th>( \hat{\beta}_4 )</th>
<th>( \hat{\beta}_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00</td>
<td>0.03</td>
<td>-0.03</td>
<td>0.36</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>-0.91</td>
<td>-0.03</td>
<td>-0.46</td>
<td>-2.40</td>
<td>-1.45</td>
<td></td>
</tr>
<tr>
<td>-0.93</td>
<td>0.10</td>
<td>0.41</td>
<td>1.12</td>
<td>-0.48</td>
<td></td>
</tr>
<tr>
<td>-3.38</td>
<td>1.00</td>
<td>1.00</td>
<td>-0.41</td>
<td>2.28</td>
<td></td>
</tr>
<tr>
<td>-1.89</td>
<td>-0.93</td>
<td>-1.03</td>
<td>2.98</td>
<td>0.76</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Weights</th>
<th>( \hat{\alpha}_{0,1} )</th>
<th>( \hat{\alpha}_{0,2} )</th>
<th>( \hat{\omega}_3 )</th>
<th>( \hat{\omega}_4 )</th>
<th>( \hat{\omega}_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.07</td>
<td>0.04</td>
<td>-0.01</td>
<td>0.00</td>
<td>-0.01</td>
<td></td>
</tr>
<tr>
<td>-0.02</td>
<td>0.00</td>
<td>-0.04</td>
<td>0.01</td>
<td>0.01</td>
<td></td>
</tr>
<tr>
<td>0.10</td>
<td>-0.01</td>
<td>-0.15</td>
<td>-0.04</td>
<td>-0.05</td>
<td></td>
</tr>
<tr>
<td>0.03</td>
<td>-0.15</td>
<td>0.03</td>
<td>0.01</td>
<td>-0.02</td>
<td></td>
</tr>
<tr>
<td>0.06</td>
<td>0.29</td>
<td>0.01</td>
<td>0.03</td>
<td>-0.01</td>
<td></td>
</tr>
</tbody>
</table>

substantially different from the long-run, the two graphs will usually look quite different. This illustrates the importance of specifying the short-run as well as the long-run, even if the main interest is in the long-run. For instance, if the interest is only in the PPP relation, it would be tempting to restrict the basic variable set to \( \{ p_1, p_2, e_{1,t} \} \). But this would result in a model where the important short-run effects from the asset markets measured by the interest rates would not be accounted for, thus possibly invalidating the estimation of the long-run PPP.

4.4. The unrestricted cointegration space

Since the test statistics and the graphical examination supported the choice of \( r = 2 \), the subsequent analysis will be based on the assumption of two stationary relations and three common trends in this data set. We now turn to the analysis of the individual estimates which are given in table 3.

Before commenting on the individual estimates it should be pointed out that any linear combination of the stationary vectors is also a stationary vector and therefore a direct interpretation is not always interesting. This points to the need to use testing as a device to find out whether any specified structural relation can be contained in the space spanned by \( \beta \). However, in this case the first eigenvector seems to contain the assumed PPP relation among the first three variables and the second seems to contain the interest
rate differential among the last two variables. The $\hat{a}_{ij}$ coefficients seem to indicate that the first eigenvector is most important for the UK price and the UK effective exchange rate equations, whereas the coefficient $\hat{a}_{21}$ in the combined price equation is essentially zero. The $\alpha_{i2}$ coefficients indicate that the interest rate differential is important only in the two interest rate equations. However, formal testing would be needed to make these statements more precise. The joint hypothesis $\{\alpha_i = 0\}$ can be tested by the likelihood ratio test procedure described in Johansen (1991a) and Johansen and Juselius (1990). Testing each $\alpha_{ij}$ individually is also possible, but reasonable only under the assumption that the estimated $\beta_i$ vectors (based on the normalization $\beta S_{kk} \beta' = I$) are the cointegrating vectors of interest instead of linear combinations of these. Since this is not necessarily the case we will in section 4.5 only test the hypothesis $\{\alpha_{2j} = 0, j = 1, 2\}$ and $\{\alpha_{3j} = 0, j = 1, 2\}$.

It is quite interesting to investigate the estimates of the restricted $\tilde{\Pi} = \hat{\alpha} \beta'$ for $r = 2$ as given in table 4. These estimates measure the combined effect of the two cointegrating relations in each of the five equations.

Table 4
The estimates of $\Pi = \alpha \beta'$ for $\mathcal{H}_r(2)$.

<table>
<thead>
<tr>
<th>Eq.</th>
<th>$p_1$</th>
<th>$p_2$</th>
<th>$e_{12}$</th>
<th>$i_1$</th>
<th>$i_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.067</td>
<td>0.061</td>
<td>0.060</td>
<td>0.272</td>
<td>0.090</td>
</tr>
<tr>
<td>2</td>
<td>-0.018</td>
<td>0.016</td>
<td>0.016</td>
<td>0.064</td>
<td>0.030</td>
</tr>
<tr>
<td>3</td>
<td>0.101</td>
<td>-0.091</td>
<td>-0.093</td>
<td>-0.345</td>
<td>-0.186</td>
</tr>
<tr>
<td>4</td>
<td>0.030</td>
<td>-0.026</td>
<td>-0.018</td>
<td>-0.263</td>
<td>0.072</td>
</tr>
<tr>
<td>5</td>
<td>0.066</td>
<td>-0.062</td>
<td>-0.087</td>
<td>0.097</td>
<td>-0.882</td>
</tr>
</tbody>
</table>

Note that the PPP relation seems to be present in all equations with the greatest weight in the exchange rate equation followed by the UK price equation and the Eurodollar rate equation. It is amazing how closely the estimates follow the hypothetical PPP $(a_i, -a_i, -a_i)$ relation, where $a_i$ is the weight coefficient in eq. $i$. Note also that the sign of $a_i$ is consistent with the expected signs for the UK equations, i.e., eq. 1, eq. 3, and eq. 4. For the combined price equation the coefficients are approximately zero, which is consistent with a hypothesis of weak exogeneity for the long-run parameters $\alpha$ and $\beta$. The coefficients to the interest rates are also quite close to the expected ones. However, in the first three equations the weighted sum of the interest rates instead of the interest rate differential seems to be relevant, even though the coefficient to the Eurodollar rate is small enough to be statistically insignificant in the price equations. The same result was found in a similar study of prices, exchange rates, and interest rates between Denmark and Germany [Juselius (1991)] which makes the observation more interesting.
Table 5

<table>
<thead>
<tr>
<th>Eigenvalues $\hat{\lambda}_i$</th>
<th>Test statistics $-T \ln(1 - \hat{\lambda}_i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{H}_1$</td>
<td></td>
</tr>
<tr>
<td>0.407</td>
<td>31.3</td>
</tr>
<tr>
<td>0.285</td>
<td>20.2</td>
</tr>
<tr>
<td>0.254</td>
<td>17.6</td>
</tr>
<tr>
<td>0.102</td>
<td>6.5</td>
</tr>
<tr>
<td>0.083</td>
<td>5.2</td>
</tr>
<tr>
<td>$\mathcal{H}_2$</td>
<td></td>
</tr>
<tr>
<td>0.400</td>
<td>30.6</td>
</tr>
<tr>
<td>0.277</td>
<td>19.5</td>
</tr>
<tr>
<td>0.158</td>
<td>10.3</td>
</tr>
<tr>
<td>0.088</td>
<td>5.5</td>
</tr>
</tbody>
</table>

4.4. Testing for weak exogeneity

It was shown in Johansen and Juselius (1990) that if for some $i \alpha_i = 0$, then $\Delta X_i$ is weakly exogenous for $\alpha$ and $\beta$ in the sense that the conditional distribution of $\Delta X_i$ given $\Delta X_{it}$ as well as the lagged values of $X_i$ contains the parameters $\alpha$ and $\beta$, whereas the distribution of $\Delta X_i$ given the lagged $X_i$ does not contain the parameters $\alpha$ and $\beta$. Furthermore, the parameters in the conditional and marginal distribution are variation-independent. Note however that the condition of weak exogeneity holds only for the case when the parameters of interest are the long-run parameters $\alpha$ and $\beta$. See Johansen (1991a) for a full discussion of this topic. We will first test the hypothesis

$$\mathcal{H}_2: \alpha_{2j} = 0 \text{ for } j = 1, 2,$$

which is of particular interest in this case since the Gaussian assumption about the residuals $\xi_{zt}$ was not completely satisfied due to excess kurtosis. If $\mathcal{H}_2$ is accepted, the system could be reduced to a four-dimensional system by conditioning on $\Delta x_{zt}$ without affecting the estimates of $\beta$. Solving the model under the restriction $\mathcal{H}_2$ gives rise to the $p - 1$ new eigenvalues in table 5 to be compared with the eigenvalues of the unrestricted model $\mathcal{H}_1$. The hypothesis is tested by comparing the restricted model within the unrestricted model $\mathcal{H}_1$ using the likelihood ratio test procedure derived in Johansen (1991b). This amounts to comparing the $r = 2$ first eigenvalues under $\mathcal{H}_2$ with the corresponding eigenvalues under $\mathcal{H}_1$ in the following way:

$$-2 \ln Q(\mathcal{H}_2 | \mathcal{H}_1) = 60 \ln \left\{ \frac{(1 - 0.400)(1 - 0.277)}{(1 - 0.407)(1 - 0.285)} \right\}$$

$$= 0.65 + 0.66 = 1.31.$$
weakly exogenous for $\beta$, i.e.:

$$\mathcal{H}_3: \quad \alpha_{5j} = 0 \quad \text{for} \quad j = 1, 2,$$

giving the test statistic:

$$-2 \ln Q(\mathcal{H}_3 | \mathcal{H}_1) = 60 \ln \left( \frac{(1 - 0.387)(1 - 0.231)}{(1 - 0.407)(1 - 0.285)} \right)$$

$$= 1.96 + 4.38 = 6.34.$$

The $\chi^2(2)$ test statistic is now more significant indicating that the Eurodollar rate cannot be considered weakly exogenous for $\beta$.

5. A class of tests for linear structural hypotheses on the cointegration vectors

The hypotheses are formulated in terms of the cointegrating relations $\beta$, since these describe the long-run relations in which most economic structural hypotheses are formulated. Examples and motivation are given below, but for later reference the hypotheses are the following:

$$\mathcal{H}_4: \quad \beta = H_4 \varphi, \quad H_4(p \times s), \varphi(s \times r), \quad r \leq s \leq p, \quad (14)$$

$$\mathcal{H}_5: \quad \beta = (H_5, \psi), \quad H_5(p \times r_1), \psi(p \times r_2), \quad r = r_1 + r_2, \quad (15)$$

$$\mathcal{H}_6: \quad \beta = (H_6 \varphi, \psi), \quad H_6(p \times s), \varphi(s \times r_1), \psi(p \times r_2), \quad (16)$$

$$r_1 \leq s \leq p, \quad r = r_1 + r_2.$$

These hypotheses are linear hypotheses on the cointegrating relations, which are structural in the sense that they do not depend on any normalization of the parameter $\beta$. The hypothesis $\mathcal{H}_4$ was discussed in detail in Johansen and Juselius (1990), whereas $\mathcal{H}_5$ and $\mathcal{H}_6$ are new.

It will be shown by an analysis of the likelihood function how estimators and test statistics can be calculated. The calculations are reduced to an eigenvalue problem, such that the analysis of the hypotheses under the various restrictions is similar to the one that is outlined in the beginning of section 3.
Since these hypotheses are really hypotheses about the space spanned by $\beta$, the cointegrating space, we can also formulate the hypotheses as

$$\mathcal{H}_4: \text{sp}(\beta) \subseteq \text{sp}(H_4),$$

$$\mathcal{H}_5: \text{sp}(H_5) \subseteq \text{sp}(\beta),$$

$$\mathcal{H}_6: \dim(\text{sp}(\beta) \cap \text{sp}(H_0)) \geq r_1.$$

The first hypothesis formulates the same $(p-s)$ linear restrictions on all the $r$ cointegrating relations. Here $r \leq s \leq p$, where $r-s$ means that all cointegrating relations are assumed known, and $s=p$ indicates that no restrictions are imposed, such that the hypothesis reduces to the model considered in section 3.

The next hypothesis assumes $r_1$ of the $r$ cointegrating relations known, as specified by the matrix $H_5$. The remaining $r_2$ relations ($\psi$) are chosen without restrictions. If $r_2=0$, i.e., $r=r_1$, the hypothesis $\mathcal{H}_5$ reduces to a special case of $\mathcal{H}_4$, with $s=r=r_1$. This type of hypothesis for $r_1=1$ has been widely tested using the Dickey–Fuller type of univariate testing procedure [Dickey and Fuller (1979)]. Our procedure differs in a very important respect by formulating the hypothesis of stationarity of for instance the PPP relation as the null hypothesis, whereas the usual univariate tests formulate the null of nonstationarity. This change in formulation makes it difficult to compare the two procedures. Moreover our test is a $\chi^2$ test, whereas the univariate tests are Dickey–Fuller type tests. In our likelihood-based procedure the test applied in connection with the determination of the cointegration rank in section 4.3 is a multivariate formulation of the Dickey–Fuller type, but once this has been determined further tests like the stationarity of a known relation can be performed using the usual $\chi^2$ distribution.

Finally hypothesis $\mathcal{H}_6$ puts some restrictions on $r_1$ of the cointegrating relations ($\varphi$) which are chosen in the space $H_6$ and the remaining $\psi$ are chosen without any restrictions. It follows that if $r_2=0$, then all relations are chosen in $\text{sp}(H_6)$ and the hypothesis reduces to $\mathcal{H}_4$, and if $r_1=s$, then $\mathcal{H}_6$ reduces to $\mathcal{H}_5$.

It was shown in Johansen (1991b) that the asymptotic distribution of the maximum likelihood estimator for $\beta$ is a mixture of Gaussian distributions. This implies that hypotheses as the ones considered give rise to a likelihood ratio test that is asymptotically distributed as $\chi^2$. We shall apply this general result in the following, and concentrate on the derivation and interpretation of the test statistics. In each case we calculate the degrees of freedom for the test. The models $\mathcal{H}_4$, $\mathcal{H}_5$, and $\mathcal{H}_6$ are tested against $\mathcal{H}_1$. Throughout the estimate under the model $\mathcal{H}_1$ is denoted by $\hat{\cdot}$ and under any of the submodels by $\hat{\cdot}$. 
5.1. The hypothesis $\beta = H_4\varphi$

This hypothesis where $H_4(p \times s)$ is known and $\varphi(s \times r)$ is unknown was treated in Johansen (1988), Johansen and Juselius (1990), and Johansen (1991b).

Since the hypothesis only involves the parameters in $\beta$, it is convenient to use eq. (5) which for $\beta = H_4\varphi$ becomes:

$$R_{0t} = \alpha \varphi' H'_4 R_{kt} + \text{error.}$$

This immediately shows that the solution is similar to that of (5), only with $R_{kt}$ replaced by $H'_4 R_{kt}$. Thus the estimation procedure is the same, only the levels are transformed into the set of variables where cointegration is to be found.

This shows that the estimator under $H_3, \hat{\varphi}$, is found as the eigenvectors of the equation

$$\lambda H'_4 S_{kk} H_4 - H'_4 S_{kj} S_{00}^{-1} S_{0k} H_4 = 0. \quad (18)$$

This equation has solutions $\lambda_1 > \cdots > \lambda_s > 0$ and eigenvectors $\hat{\nu} = (\hat{v}_1, \ldots, \hat{v}_s)$. The estimate of $\varphi$ is then $(\hat{v}_1, \ldots, \hat{v}_s)$ such that

$$\hat{\beta} = H_4(\hat{v}_1, \ldots, \hat{v}_s) \quad (19)$$

and

$$L_{\max}^{-2/T} = |S_{00}| \prod_{i=1}^{s} (1 - \hat{\lambda}_i). \quad (20)$$

From this we find the first result:

**Theorem 1.** The hypothesis $\beta = H_4\varphi$, where $H_4$ is $p \times s$, can be tested by the likelihood ratio test given by

$$-2 \ln Q(H_4|H_1) = T \sum_{i=1}^{r} \ln \left\{ \psi(1 - \hat{\lambda}_i)/(1 - \tilde{\lambda}_i) \right\}, \quad (21)$$

which is asymptotically distributed as $\chi^2$ with $f = (p - s)r$ degrees of freedom. The estimate of $\beta$ is found from (19) and $\alpha$ from (9).

**Proof.** The result follows from the above derivation. The degrees of freedom are calculated as follows: Normalize the $\beta$ matrix such that $\beta' = (I, \tau')$, with $\tau(p - r) \times r$. This gives $pr + (p - r)r$ free parameters under $H_4$. Under
Table 6

The eigenvalues under the unrestricted model $\mathcal{H}_1$ and the restricted model $\mathcal{H}_{4,1}$.

<table>
<thead>
<tr>
<th>Eigenvalues $\lambda_i$</th>
<th>Test statistics $-T \ln(1 - \lambda_i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{H}_1$</td>
<td>0.407 0.285 0.254 0.102 0.083</td>
</tr>
<tr>
<td>$\mathcal{H}_{4,1}$</td>
<td>0.386 0.278 0.090</td>
</tr>
</tbody>
</table>

the restriction (14), where $\beta = H_4 \phi$, $\phi$ is normalized in the same way, leaving $pr + (s - r)r$ free parameters under the hypothesis. The difference is the degrees of freedom for the test.

Two hypotheses of type (14) are relevant for our empirical problem. The first one is the hypothesis of the purchasing power parity formulated as: The variables $p_1$, $p_2$, and $e_{12}$ enter into the cointegrating relations with coefficients proportional to $(1, -1, -1)$, i.e., the cointegration relations are of the form $(a_i, -a_i, -a_i, *, *, *)$ for $i = 1, \ldots, r$. This can be formulated as a hypothesis of the type (14) with

$$H_{4,1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$ 

The solution to the eigenvalue problem (18) with $H_4 = H_{4,1}$ and $S_{kk}$, $S_{k0}$, and $S_{00}$ as defined by (4) is the $s = 3$ eigenvalues reported in table 6. These are compared with the eigenvalues from the unrestricted model $\mathcal{H}_1$ with $r = 2$. For $r = 2$ the likelihood ratio test (21) becomes:

$$-2 \ln Q(\mathcal{H}_{4,1} | \mathcal{H}_1) = 60 \ln \left( \frac{(1 - 0.386)(1 - 0.278)}{(1 - 0.407)(1 - 0.285)} \right)$$

$$= 2.09 + 0.59 = 2.68.$$ 

The test statistic is asymptotically distributed as $\chi^2(4)$ and thus this hypothesis cannot be rejected. This is of course consistent with the observation that the estimated coefficients in $\hat{H} = \hat{\alpha} \hat{\beta}$ for $r = 2$ closely approximated the PPP restriction in all equations of the system. Note however that the PPP restriction is not valid for the third vector $\hat{e}_3$ as can be seen by the drop in the $-T \ln(1 - \lambda_3)$ from 17.6 to 5.6. This is intuitively reasonable since $\hat{e}_3$ has been found to lie in the nonstationarity space. If we erroneously would have included the third vector among the stationary relations, the beautiful PPP structure found in $\hat{H} = \hat{\alpha} \hat{\beta}$ for $r = 2$ would have been diffused by the third
Table 7
The eigenvalues under the unrestricted model $\mathcal{H}_1$ and the restricted model $\mathcal{H}_{4,2}$.

<table>
<thead>
<tr>
<th>Eigenvalues $\hat{\lambda}_i$</th>
<th>Test statistics $-T \ln(1 - \hat{\lambda}_i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{H}_1$</td>
<td>0.407 0.285 0.254 0.102 0.083 31.3 20.2 17.6 6.5 5.2</td>
</tr>
<tr>
<td>$\mathcal{H}_{4,2}$</td>
<td>0.286 0.254 0.146 0.093 20.2 17.6 9.5 5.8</td>
</tr>
</tbody>
</table>

This serves as an illustration of how careful one has to be not to get misleading results because of the complicated interaction between stationary and nonstationary processes, as well as between short-run and long-run dynamics. It also illustrates a methodological point, namely that this test procedure has the property that once the calculation of eigenvectors has been performed, one can conduct inference for different values of $r$ without recalculating estimates and test statistics.

The second hypothesis of interest states that only the nominal interest differential enters all cointegration relations. This can be formulated as a hypothesis of type (14) with $H_4 = H_{4,2}$,

$$H_{4,2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$  

The solution to the eigenvalue problem (18) with $H_4 = H_{4,2}$ gives the $s = 4$ eigenvalues reported in table 7, where again $\mathcal{H}_{4,2}$ is compared with $\mathcal{H}_1$ for $r = 2$. In this case the likelihood ratio test statistic becomes 13.17 to be compared to a $\chi^2(2)$ distribution. The hypothesis $\mathcal{H}_{4,2}$ is thus strongly rejected. Note that the main contribution to the test statistic comes from the first eigenvector. This is a consequence of the fact already commented on that in the unrestricted model the interest rates enter the first vector with equal signs.

Given the results of the tests so far it might be of interest to test whether the PPP relation and the nominal interest rate differential are stationary processes by themselves, i.e., to test whether $[1, -1, -1, 0, 0]'X_t$ and $[0, 0, 0, 1, -1]'X_t$ are stationary. This is the next structural test to be discussed below.

5.2. The hypothesis $\beta = (H_5, \psi)$

In this hypothesis $H_5$ is a known $(p \times r_1)$ matrix and $\psi(p \times r_2)$ is unknown ($r = r_1 + r_2$). Thus $r_1$ relations are assumed known and the remaining $r_2 = r - r_1$ are to be estimated independently of the vectors in $H_5$. We split $\alpha$
accordingly into $\alpha = (\alpha_1, \alpha_2)$ so that (5) now becomes

$$R_{0,t} = \alpha_1 H_5' R_{kt} + \alpha_2 \psi' R_{kt} + \text{error}. \quad (22)$$

This model is easily estimated by first concentrating the model with respect to $\alpha_1$ by regression. Thus, if we assume that $H_5' X_t$ is stationary, we start the analysis by regressing on the stationary components, just as we regress on the variables $\Delta X_{t-1}, \ldots, \Delta X_{t-k+1}$. This gives new residuals $R_{0,ht}$ and $R_{k,ht}$, and the concentrated likelihood function gives rise to the reduced rank regression problem

$$R_{0,ht} = \alpha_2 \psi' R_{k,ht} + \text{error}. \quad (23)$$

The eq. (23) has the same form as (5) and can hence be solved by the same minimization procedure, i.e.,

$$|\psi'(S_{kk,h} - S_{k0,h} S_{00,h}^{-1} S_{0k,h})\psi| / |\psi' S_{kk,h} \psi|$$

has to be minimized over all $p \times r_2$ matrices $\psi$. Here

$$S_{ij,h} = S_{ij} - S_{ik} H_5 (H_5' S_{kk} H_5)^{-1} H_5' S_{kj}, \quad i,j=0,k.$$ 

Note that $S_{kk,h}$ is of rank $p - r_1$ and hence singular, but that $S_{k0,h} S_{00,h}^{-1} S_{0k,h}$ has the same singularity, hence Lemma 1 proved in appendix 3 shows how to minimize, by first diagonalizing $S_{kk,h}$, and then reduce $S_{k0,h} S_{00,h}^{-1} S_{0k,h}$ by the eigenvectors of $S_{kk,h}$ to a $(p - r_1) \times (p - r_1)$ matrix whose eigenvectors determine the estimate of $\psi$. Thus we first solve the eigenvalue problem

$$|\tau I - S_{kk,h}| = 0,$$

and pick out the eigenvectors corresponding to the $p - r_1$ positive eigenvalues and define $C = (e_1, \ldots, e_{p-r_1}) \text{diag}(\tau_1^{-1/2}, \ldots, \tau_{p-r_1}^{-1/2})$. Next we solve the eigenvalue problem

$$|\lambda I - C'S_{k0,h} S_{00,h}^{-1} S_{0k,h} C| = 0. \quad (24)$$

This equation has solutions $\hat{\lambda}_1 > \cdots > \hat{\lambda}_{p-r_1} > 0$ and eigenvectors $\hat{V} = (\hat{v}_1, \ldots, \hat{v}_{p-r_1})$. Thus $\hat{\psi} = C(\hat{v}_1, \ldots, \hat{v}_{r_2})$ such that

$$\hat{\beta} = (H_5, \hat{\psi}) \quad (25)$$
and

\[ L_{\text{max}}^{-2/T} = |S_{00,h}| \prod_{i=1}^{r_2} (1 - \hat{\lambda}_i) \]  

(26)

It gives a convenient formulation to solve the eigenvalue problem

\[ |\rho H_5 S_{kk} H_5 - H_5 S_{00}^{-1} S_{0k} H_5| = 0 \]  

(27)

for the eigenvalues \( \hat{\rho}_1 > \cdots > \hat{\rho}_{r_1} \), since we can then write

\[ |S_{00,h}| = |S_{00}| |H_5 (S_{kk} - S_{00}^{-1} S_{0k}) H_5| / |H_5 S_{kk} H_5| \]

\[ = |S_{00}| \prod_{i=1}^{r_1} (1 - \hat{\rho}_i). \]  

(28)

In the present context we have that \( \hat{\alpha}_i S_{00,h} \hat{\alpha}_2 = \text{diag}(\hat{\lambda}_1, \ldots, \hat{\lambda}_{r_1}) \), such that \( \hat{\lambda}_i \) measures the size of the coefficients to the cointegrating relations with respect to the matrix \( S_{00,h} \). However, an intuitive interpretation of \( \hat{\rho}_i \) similar to that of \( \hat{\lambda}_i \) is not possible. Combining the above results we then get:

**Theorem 2.** The hypothesis \( \mathcal{H}_5 : \beta = (H_5, \psi) \) can be tested by the likelihood ratio test

\[ -2 \ln Q(\mathcal{H}_5 | \mathcal{H}_1) \]

\[ = T \left( \sum_{i=1}^{r_1} \ln(1 - \hat{\rho}_i) + \sum_{i=1}^{r_2} \ln(1 - \hat{\lambda}_i) - \sum_{i=1}^{r} \ln(1 - \hat{\lambda}_i) \right), \]  

(29)

which is asymptotically distributed as \( \chi^2 \) with \( f = (p - r) r_1 \) degrees of freedom. The estimate of \( \beta \) is found from (25) and \( \alpha \) is given by (6).

**Proof.** We shall calculate the degrees of freedom. We again normalize \( \beta \) as before with \( \tau \) of dimension \((p - r) \times r\). Now fixing the first \( r_1 \) columns of \( \tau \) amounts to fixing \( r_1 (p - r) \) parameters, hence the degrees of freedom \( r(p - r) - r_2 (p - r) = r_1 (p - r) \).

The first hypothesis, of the type \( \mathcal{H}_5 \), we shall consider here is whether the PPP relation is stationary on its own. Let \( H_{5:1} = [1, -1, -1, 0, 0]' \) in (15). The solution of (24) gives us an eigenvalue \( \hat{\lambda}_1 \), since \( r_2 = 1 \), corresponding to the solution of (24) and, since \( r_1 = 1 \), an estimate \( \hat{\rho}_1 \) defined as the solution of (27). The results are given in table 8.
The likelihood ratio test (29) becomes:

\[-2 \ln Q(\mathcal{H}_{5,1} \mid \mathcal{H}_1) = 60 \ln \left( \frac{(1 - 0.396)(1 - 0.106)}{(1 - 0.407)(1 - 0.285)} \right) = 14.53.\]

The test statistic is asymptotically distributed as \(\chi^2(3)\) and the hypothesis that PPP relation is stationary by itself is rejected.

The second hypothesis of interest \(\mathcal{H}_{5,2}\) is whether the interest rate differential is stationary. We take \(H_{5,2} = [0, 0, 0, 1, -1]'\) in (24) and get the eigenvalues of Table 9.

The likelihood ratio test statistic becomes:

\[-2 \ln Q(\mathcal{H}_{5,2} \mid \mathcal{H}_1) = 60 \ln \left( \frac{(1 - 0.406)(1 - 0.263)}{(1 - 0.407)(1 - 0.285)} \right) = 1.93.\]

The test statistic is asymptotically distributed as \(\chi^2(3)\) and thus not significant. We conclude that the interest rate differential by itself is a stationary process.

Based on the test results of this section one might ask whether there exists another linear combination between \(p_1\), \(p_2\), and \(e_{12}\) that is stationary. This type of structural hypothesis will be considered in the next section.
5.3. The hypothesis $\beta = (H_6 \varphi, \psi)$

As before we start from (5) which in this case has the form

$$R_{0t} = \alpha_1 \varphi' H_6' R_{kt} + \alpha_2 \psi' R_{kt} + \text{error},$$

where $\alpha_1(p \times r_1)$, $\varphi(s \times r_1)$, and $\psi(p \times r_2)$ have to be estimated and $H_6(p \times s)$ is known. Here $r_1 \leq s \leq p - r_2$. This problem does not as easily reduce to an eigenvalue problem, but instead we shall apply a simple switching algorithm to maximize the likelihood function. The algorithm is described briefly as follows:

1. For fixed $\varphi$ concentrate with respect to $\alpha_1$ by regression and then solve the reduced rank problem for $\alpha_2$ and $\psi$.

2. Now fix $\psi$, concentrate the likelihood function with respect to $\alpha_2$, and solve the reduced rank problem for $\alpha_1$ and $\varphi$.

3. Repeat step (1) and (2) until convergence.

Since the likelihood function is maximized at each step and since $\beta$ can be restricted to a compact set by the normalization $\beta' S_{kk} \beta = I$, the algorithm does converge, but it could be to a local maximum. We do not have a proof that the algorithm converges to the global maximum, but in the cases we have used it there has been no problems and convergence was attained after a few steps from different starting values. The algorithm is based on Lemma 1 given in appendix 3. To get started choose $\psi = 0$ and $\varphi$ to solve the eigenvalue problem

$$|\lambda H_6' S_{kk} H_6 - H_6' S_{k0} S_{00}^{-1} S_{0k} H_6| = 0$$

for $\lambda_1 > \cdots > \lambda_s > 0$ and $(\hat{\varphi}_1, \ldots, \hat{\varphi}_s)$, such that $\hat{\beta}_1 = H_6 \hat{\varphi} = H_6(\hat{\varphi}_1, \ldots, \hat{\varphi}_s)$.

The first step of the algorithm consists of fixing this value of $\beta_1$, concentrate with respect to $\alpha_1$, i.e., condition on $\hat{\beta}_1' R_{kt}$, and find the $\psi$ that minimizes

$$\left| \psi' (S_{kk, \hat{\beta}_1} - S_{k0, \hat{\beta}_1} S_{00, \hat{\beta}_1}^{-1} S_{0k, \hat{\beta}_1}) \psi \right| / \left| \psi' S_{kk, \hat{\beta}_1} \psi \right|. \quad (31)$$

The matrix $S_{kk, \hat{\beta}_1}$ is singular, since $S_{kk, \hat{\beta}_1} \hat{\beta}_1 = 0$, but $S_{k0, \hat{\beta}_1} S_{00, \hat{\beta}_1}^{-1} S_{0k, \hat{\beta}_1}$ is singular with the same null space. Hence Lemma 1 shows how one can find the eigenvalues $\lambda_1, \ldots, \lambda_{p - r_1}$ and eigenvectors $\hat{\varphi}_1, \ldots, \hat{\varphi}_{p - r_1}$ such that $\hat{\beta}_2 = (\hat{\varphi}_1, \ldots, \hat{\varphi}_{r_2})$. 
The second step of the algorithm is to fix \( \hat{\beta}_2 \), concentrate with respect to \( \alpha_2 \), and determine a new estimate of \( \hat{\beta}_1 \) by minimizing

\[
\left| \varphi^t H_0 \left( S_{kk, \hat{\beta}_2} - S_{k0, \hat{\beta}_2} S_{00, \hat{\beta}_2}^{-1} S_{0k, \hat{\beta}_2} \right) H_0 \varphi \right| \left/ \left| \varphi^t H_0 S_{kk, \beta} H_0 \varphi \right| \right. ,
\]

(32)

This problem can again be solved by the procedure in Lemma 1, even though the matrix \( H_0 S_{kk, \hat{\beta}_2} H_0 \) is nonsingular. This gives optimal eigenvalues \( \hat{\omega}_1, \ldots, \hat{\omega}_r \) and eigenvectors \( \hat{\nu}_1, \ldots, \hat{\nu}_r \), such that \( \hat{\beta}_1 = H_0 (\hat{\nu}_1, \ldots, \hat{\nu}_r) \). The maximized likelihood function has two expressions corresponding to the two steps of the algorithm:

\[
L_{\text{max}}^{-2/\ell_T} = |S_{00, \hat{\beta}_1}| \prod_{i=1}^{r_1} (1 - \hat{\lambda}_i) = |S_{00, \hat{\beta}_2}| \prod_{i=1}^{r_1} (1 - \hat{\omega}_i),
\]

(33)

where \( \hat{\beta}_1 = H_0 \hat{\phi} \) and \( \hat{\beta}_2 = \hat{\psi} \). Again the maximized likelihood can be given the different expression by solving the eigenvalue problem

\[
|\rho \hat{\beta}_1 S_{kk} \hat{\beta}_1 - \hat{\beta}_1 S_{k0} S_{00}^{-1} S_{0k} \hat{\beta}_1| = 0
\]

for the eigenvalues \( \hat{\rho}_1 > \cdots > \hat{\rho}_r \), since we can then write

\[
|S_{00, \hat{\beta}_1}| = |S_{00}| \prod_{i=1}^{r_1} (1 - \hat{\rho}_i).
\]

The results above can be summarized in:

Theorem 3. The hypothesis \( \mathcal{H}_0; \beta = (H_0 \varphi, \psi) \) can be tested by a likelihood ratio test of the form

\[
-2 \ln Q(\mathcal{H}_0, \mathcal{H}_1)
\]

\[
= T \left\{ \sum_{i=1}^{r_1} \ln(1 - \hat{\rho}_i) + \sum_{i=1}^{r_2} \ln(1 - \hat{\lambda}_i) - \sum_{i=1}^{r} \ln(1 - \hat{\lambda}_i) \right\},
\]

(34)

which is asymptotically distributed as \( \chi^2 \) with \( f = (p - s - r_2) r_1 \) degrees of
Table 10
The eigenvalues under the unrestricted model $\mathcal{H}_1$ and the restricted model $\mathcal{H}_0$.

<table>
<thead>
<tr>
<th>$\mathcal{H}_1 \lambda$</th>
<th>$\mathcal{H}_0 \hat{\lambda}$</th>
<th>$\hat{\rho}_1$</th>
<th>Test statistics $-T \ln(1 - \hat{\lambda})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.407</td>
<td>0.407</td>
<td>0.256</td>
<td>31.3 20.2 17.6 6.5 5.2</td>
</tr>
<tr>
<td>0.285</td>
<td>0.284</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.254</td>
<td>0.102</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.102</td>
<td>0.083</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.083</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

freedom. The estimator is calculated by a switching algorithm by solving successive minimization problems (31) and (32).

Proof. The proof follows from the above calculations. We here derive the degrees of freedom for the test. Let us write the parameters as $\alpha_1 \phi'$ and $\alpha_2 \psi'$. The parameter $\phi$ can be normalized such that the first set of parameters contains $pr_1 + r_1(s - r_1)$ free parameters. In the second set of parameters we first normalize $\psi$ to contain $r_2(p - r_2)$ parameters. These are not free since at each step of the algorithm, the parameter $\psi$ is chosen orthogonal to $\hat{\beta}_1$ which restricts the variation to a $(p - r_1)$-dimensional space.

Thus we have $pr_2 + (p - r_1 - r_2)r_2$ parameters in the second set of parameters. This gives a total of $pr + (p - r)r_2 + (s - r_1)r_1$. Subtracting this from $pr + (p - r)r$ gives the degrees of freedom.

A hypothesis of this type can be formulated by asking if there is a vector of the form $(a, b, c, 0, 0)$ in the cointegration space for some $a$, $b$, and $c$. In matrix formulation, we can define

$$H_0 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},$$

and formulate the hypothesis as

$$\mathcal{H}_0: \quad \beta = (H_0 \phi, \psi).$$

In our empirical example, $r = 2$, $r_1 = 1$, and $r_2 = 1$. The iterated solution to the eigenvalue problems (31) and (32) are given in table 10 and compared with the unrestricted results.
The likelihood ratio test (34) becomes:

$$-2 \ln Q(\mathcal{H}_0 | \mathcal{H}_1) = 60 \ln \left( \frac{(1 - 0.256)(1 - 0.407)}{(1 - 0.407)(1 - 0.285)} \right) = 2.4.$$  

The test statistic is asymptotically distributed as $\chi^2(1)$ and not significant suggesting that there might exist a linear combination between $p_1$, $p_2$, and $\epsilon_{12}$ that is stationary.

6. Concluding remarks

In this paper some likelihood ratio tests are developed to test structural hypotheses on the cointegration space in a multivariate cointegration model. It is demonstrated how the multivariate analysis in combination with the hypothesis of cointegration allows a precise formulation of a number of interesting economic hypotheses in such a way that they can be tested. The importance of these tests is illustrated by an application to the purchasing power parity and the uncovered interest rate parity relation for UK versus a trade weighted foreign country. In a five-dimensional system of equations (two prices, exchange rates, and two interest rates) we ask the question whether the PPP relation is stationary by itself, i.e., without the interest rates and correspondingly whether the nominal interest rate differential is a stationary process. The answer is negative for the PPP relation, but positive for the interest rate differential. We also ask the question whether the PPP relation with some combination of the two interest rates is stationary and find that this hypothesis can indeed be accepted. Finally, we ask the question whether there exists any linear combination between the prices and the exchange rates that is stationary. In addition one can of course make a joint test of the two accepted hypotheses, namely that the interest rate differential is stationary, $\mathcal{H}_{5.2}$, and that the PPP restriction is present in the two cointegration relations, $\mathcal{H}_{4.1}$, but this would require a slight extension of the present procedures.

Thus the empirical results seem to indicate that evidence on the PPP relation can be found by accounting for the interaction between the goods and the asset markets, but not as usually assumed for instance in the monetary model by relating real exchange rates to the interest rate differential. Instead, the results seem to indicate that the movements in real PPP exchange rates are counteracted by the movements in the level of interest rates. This seems to indicate that the determination of prices, interest rates, and exchange rates should also be investigated in a balance-of-payments framework with interrelated movements in the current account and the capital account.
Appendix 1. Graphs of the differences of the data $\Delta X_t = (\Delta p_1, \Delta p_2, \Delta e_{12},\Delta i_1,\Delta i_2)$ and the corresponding residuals from eq. (13)
Effective exchange rate

Residuals

Treasury bill rate

Residuals
S. Johansen and K. Juselius, Testing structural hypotheses

Eurodollar rate

The world price of oil

Residuals
Appendix 2. Graphs of the cointegration relations $\beta'X$, [cf. eq. (12)] and $\beta'R_k$, [cf. eq. (5)]
S. Johansen and K. Juselius, Testing structural hypotheses

Fig. B1. (continued)
S. Johansen and K. Juselius, Testing structural hypotheses

Cointegration relation $S$

Beta $A_t$

The assumed relations:
- Real GNP exchange rate
- Interest rate difference
Appendix 3

The following results show that the classical result about minimizing the ratio of determinants of quadratic forms also holds if the matrices are singular in a suitable sense:

**Lemma 1.** Let $A$ and $B$ be $p \times p$ positive semidefinite matrices, such that $A$ has rank $m$ and such that $Ax = 0$ implies that $Bx = 0$ for $x \in \mathbb{R}^p$.

The expression

$$f(\beta) = \|\beta'(A - B)\beta\|/\|\beta'A\beta\|, \quad \beta'A > 0,$$

is minimized among all $p \times r$ matrices $\beta$ ($r \leq m$) by first solving the eigenvalue problem $|\rho I - A| = 0$ for $\rho_1 \geq \cdots \geq \rho_m > \rho_{m+1} = \cdots = \rho_p = 0$ and eigenvectors $(e_1, \ldots, e_p)$, and define the $p \times m$ matrix

$$C = (e_1, \ldots, e_m)\text{diag}(\rho_1^{-1/2}, \ldots, \rho_m^{-1/2}),$$

such that $C'AC = I_{m \times m}$. Next solve the reduced eigenvalue problem $|\hat{\lambda} I - C'BC| = 0$ for $\lambda_1 \geq \cdots \geq \lambda_m$ and eigenvectors $u_1, \ldots, u_m$. The solution of the minimization problem is then given by $\hat{\beta} = C(u_1, \ldots, u_r)$ and the minimum value of $f$ is $f(\hat{\beta}) = \prod_{i=1}^r (1 - \hat{\lambda}_i)$. The solution $\hat{\beta}$ is orthogonal to the null space of $A$.

**Proof.** From $A = \text{Ediag}(\rho_1, \ldots, \rho_p)E'$ it follows, since $\rho_{m+1} = \cdots = \rho_p = 0$, that

$$\beta'A\beta = \beta'(e_1, \ldots, e_m)\text{diag}(\rho_1, \ldots, \rho_m)(e_1, \ldots, e_m)'\beta.$$ 

Thus $\beta'A\beta$ only depends on $\beta$ through its projection onto $(e_1, \ldots, e_m)$. This also holds for $A - B$, since $Ae_i = 0$ implies that $Be_i = 0$, and hence it also holds for $f(\beta)$. Thus we reduce the problem to the $m$-dimensional space spanned by $(e_1, \ldots, e_m)$ and solve it there using the classical tools from multivariate analysis [see Rao (1973)].

The orthogonality of $\hat{\beta}$ to the null space of $A$ follows from $\hat{\beta} \in \text{sp}(e_1, \ldots, e_m)$ and that the null space is spanned by $(e_{m+1}, \ldots, e_p)$.

What is happening here is roughly the following: For nonsingular matrices the above minimization problem is solved by solving the eigenvalue problem:

$$|\hat{\lambda} A - B| = 0.$$

When $A$ and also $B$ are singular in the way described, the determinant is identically zero for all values of $\hat{\lambda}$, and hence not very helpful for the solution. Hence one considers the equation reduced to the $m$-dimensional orthogonal complement of the null space of $A$, because $A$ is nonsingular on this space. This is what is accomplished by multiplying by $C'$ and $C$. 

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